TEICHMÜLLER DILATIONS OF VARYING HUES: FROM COMPLEX TEICHMÜLLER THEORY TO IUT TO GT [JOINT WORK IN PROGRESS WITH TSUJIMURA/TSUJIMURA-SAÏDI] (2025 ICMS VERSION)

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https://www.kurims.kyoto-u.ac.jp/~motizuki/Tch%20Dilations%20of%20Varying%20Hues%20(ICMS%202025%20version).pdf

§1. **Ring-theoretic interpretation of complex Teichmüller theory**

· Review of classical complex Teichmüller theory:
(cf. [EssLgc], Example 3.3.1)

Recall the most *fundamental deformation of complex structure* in classical complex Teichmüller theory: for $\lambda \in \mathbb{R}_{>1}$,

$$\Lambda : \mathbb{C} \rightarrow \mathbb{C}$$

$$\mathbb{C} \ni z = x + iy \mapsto \zeta = \xi + i\eta \stackrel{\text{def}}{=} \lambda \cdot x + iy \in \mathbb{C}$$

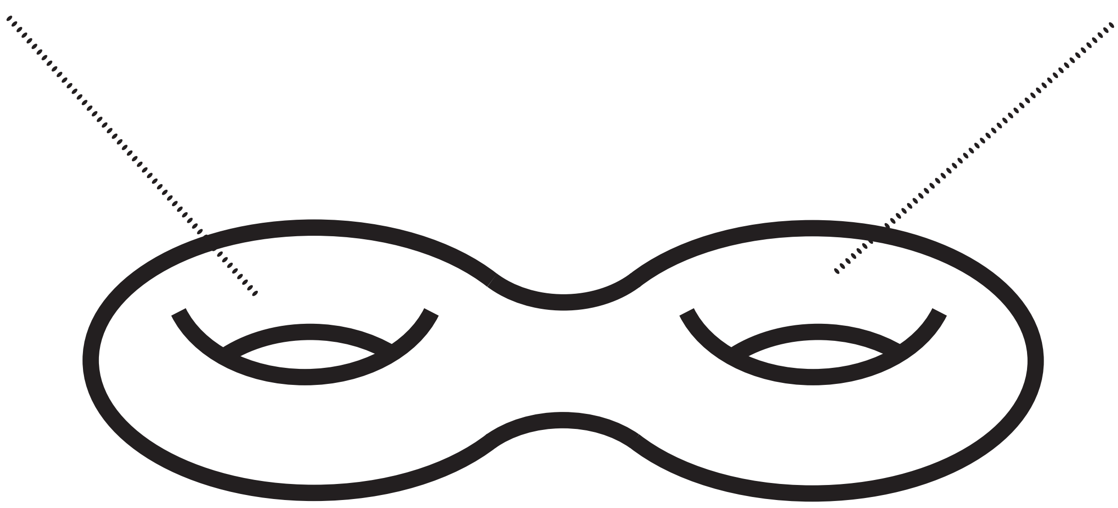
— where $x, y \in \mathbb{R}$.

· Classical complex Teichmüller theory on Riemann surfaces:
More generally, on a *single (oriented) topological surface* S, we can start with one *holomorphic structure* $\mathfrak{H}_1$ on S and a *square differential* relative to $\mathfrak{H}_1$, then form the *Teichmüller dilation*, or *Teichmüller map*, obtained by deforming (i.e., in the fashion described above) the canonical holomorphic coordinate obtained by integrating the square root of the square differential (along paths) so as to obtain a *new holomorphic structure* $\mathfrak{H}_2$.

Next, for $i = 1, 2$, write
$\mathcal{O}^{\mathfrak{H}_i}$ for the sheaf of *holomorphic functions* on S, rel. to $\mathfrak{H}_i$;
$\mathcal{O}^{\mathfrak{R}}$ for the sheaf of (complex valued) *real analytic fns.* on S.

Here, we note that for connected open subsets $U \subseteq S$, $\mathcal{O}^{\mathfrak{R}}(U)$ is a *domain*, i.e., unlike the case with continuous or $\mathcal{C}^{\infty}$-functions. That is to say, $\mathcal{O}^{\mathfrak{R}}$ is in some sense *close* to being like $\mathcal{O}^{\mathfrak{H}_i}$, for $i = 1, 2$, but still *suff'ly large* as to allow one to obtain *embeddings* in the *common container* $\mathcal{O}^{\mathfrak{R}}$:

$$\mathcal{O}^{\mathfrak{H}_1} \quad \subseteq \quad \mathcal{O}^{\mathfrak{R}} \quad \supseteq \quad \mathcal{O}^{\mathfrak{H}_2}$$



· In the remainder of the present talk, we would like to consider various *arithmetic analogues* of the function theory discussed above in the complex case.

§2. The case of inter-universal Teichmüller theory (IUT)

· A more detailed exposition of IUT may be found in

· the *survey texts* [Alien], [EssLgc], as well as in
· the *videos/slides* available at the following URLs:
(cf. also my series of *DWANGO LECTURES* on IUT
— URLs available at request!):

`https://www.kurims.kyoto-u.ac.jp/~motizuki/ExpHoriz IUT21/WS3/ExpHorizIUT21-InvitationIUT-notes.html`

`https://www.kurims.kyoto-u.ac.jp/~motizuki/ExpHoriz IUT21/WS4/ExpHorizIUT21-IUTSummit-notes.html`

· Let R be an *integral domain* (e.g., $\mathbb{Z} \subseteq \mathbb{Q}$) equipped with the action of a *group* G, $(\mathbb{Z} \ni)\ N \geq 2$. For simplicity, assume that $N = 1 + \cdots + 1 \neq 0 \in R$; R has *no nontrivial N-th roots of unity*. Write $R^{\triangleright} \subseteq R$ for the *multiplicative monoid* $R \setminus \{0\}$. Then let us observe that the *N-th power map* on $R^{\triangleright}$ determines an *isomorphism of multiplicative monoids* equipped with actions by G

$$G \curvearrowright \quad R^{\triangleright} \quad \stackrel{\sim}{\to} \quad (R^{\triangleright})^N \ (\subseteq R^{\triangleright}) \ \curvearrowleft \quad G$$

that does *not arise* from a *ring homomorphism*, i.e., it is clearly *not compatible* with *addition* (cf. our assumption on N!).

· Let $^{\dagger}R$, $^{\ddagger}R$ be *two distinct copies* of the integral domain R, equipped with respective actions by *two distinct copies* $^{\dagger}G$, $^{\ddagger}G$ of the group G. We shall use similar notation for objects with labels "†", "‡" to the previously introduced notation. Then one may use the *isomorphism of multiplicative monoids* arising from the *N-th power map* discussed above to *glue* together

$$^{\dagger}G \curvearrowright \ ^{\dagger}R \supseteq (^{\dagger}R^{\triangleright})^N \quad \stackrel{\sim}{\leftarrow} \quad ^{\ddagger}R^{\triangleright} \subseteq \ ^{\ddagger}R \curvearrowleft \ ^{\ddagger}G$$

... where the notion of a *gluing* may be understood
· as a *quotient set* via identifications, or (preferably)
· as an *abstract diagram* (cf. graphs of groups/anabelioids!)

the *ring* $^{\dagger}R$ to the *ring* $^{\ddagger}R$ along the *multiplicative monoid* $(^{\dagger}R^{\triangleright})^N \stackrel{\sim}{\leftarrow} \ ^{\ddagger}R^{\triangleright}$. This gluing is *compatible* with the respective actions of $^{\dagger}G$, $^{\ddagger}G$ relative to the isomorphism $^{\dagger}G \stackrel{\sim}{\to} \ ^{\ddagger}G$ given by forgetting the labels "†", "‡", but, since the N-th power map is *not compatible* with *addition* (!), this isomorphism $^{\dagger}G \stackrel{\sim}{\to} \ ^{\ddagger}G$ may be regarded either as an isomorphism of ("*coric*", i.e., *invariant* with respect to the N-th power map) *abstract groups* (cf. the notion of "*inter-universality*", as discussed in [EssLgc], §3.2, §3.8!) or as an isomorphism of groups equipped with actions on certain *multiplicative monoids*, but *not* as an isomorphism of ("*Galois*" — cf. the *inner automorphism indeterminacies* of SGA1!) groups equipped with actions on *rings/fields*.

· The problem of *describing (certain portions of the) ring structure* of $^{\dagger}R$ in terms of the *ring structure* of $^{\ddagger}R$ — in a fashion that is *compatible* with the *gluing* and via a *single* algorithm that may be applied to the *common* (cf. *logical AND* $\wedge$*!*) *glued data* to reconstruct *simultaneously* (certain portions of) the ring structures of *both* $^{\dagger}R$ and $^{\ddagger}R$, up to suitable relatively mild *indeterminacies* (cf. the theory of *crystals*!) — seems (at first glance/in general) to be *hopelessly intractable* (cf. the case of $\mathbb{Z}$)!

... where we note that here, considering *portions* is important because we want to *decompose* the above diagram up into *pieces* so that we can consider *symmetry* properties involving these pieces!

One well-known elementary example: when $N = p$, working *modulo p* (cf. *indeterminacies*, analogy with *crystals*!), where there is a *common ring structure* that is *compatible* with the *p-th power map*!

Another important example: Faltings' proof of *invariance* of *height* of elliptic curves under *isogeny*, under the assumption of the existence of a *global multiplicative subspace* (cf. [ClsIUT], §1; [EssLgc], Example 3.2.1)!

... This is precisely what is *achieved in IUT* by means of the *multiradial representation for the* Θ*-pilot* via

· *anabelian geometry* (cf. the *abstract groups* $^{\dagger}G$, $^{\ddagger}G$!);
· *the p-adic/complex logarithm, theta functions*;
· *Kummer theory*, to relate *Frob.-/étale-like* versions of objects.

· *Main point*:

The *multiplicative monoid* and *abstract group* structures (but *not* the ring structures!) are *common* (cf. "*logical AND* $\wedge$*!*") to †, ‡ and can be used as the *input data* for an algorithm to construct the *multirad. rep. for the* Θ*-pilot*, i.e., a *common container* for the distinct *ring strs.* (i.e., "*arith. hol. strs.*") †, ‡

$$^{\dagger}R \quad \subseteq \quad \left(\text{multirad. rep. for the } \Theta\text{-pilot}\right) \quad \supseteq \quad {}^{\ddagger}R$$

· When $R = \mathbb{Z}$ (or, in fact, more generally, the *ring of integers* "$\mathcal{O}_F$" of a number field F — cf. the multiplicative *norm map* $\mathrm{N}_{F/\mathbb{Q}} : F \to \mathbb{Q}$), one may consider the "*height/log-volume*"

$$\log(|x|) \in \mathbb{R}$$

for $0 \neq x \in \mathbb{Z}$. Then the *N-th power map* of (i), (ii) corresponds, after passing to *heights*, to *multiplying real numbers by N*; the *multiradial algorithm* corresponds to saying that the height is *unaffected (up to a mild error term!)* by *multiplication by N*, hence that the *height is bounded*!

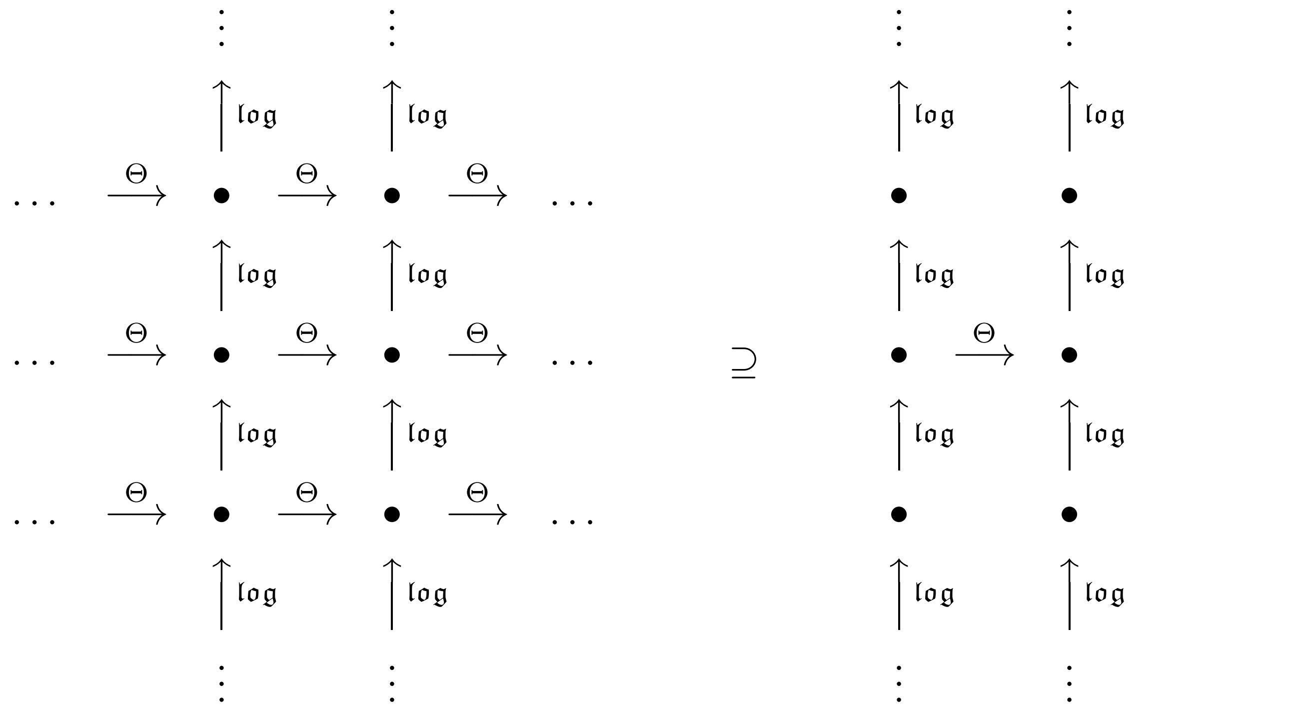
· In the case of IUT, the *multirad. rep. for the Θ-pilot* is obtained by means of a sort of "*analytic continuation*" along a certain "*infinite H*" of the *log-theta-lattice* [cf. the discussion surrounding [EssLgc], §3.3, (InfH)]

... where

· the *Θ-link* between distinct ring strs. "•" corresponds to the N-th power map discussed in the present §2, while

· the *$\mathfrak{log}$-link* locally at nonarchimedean valuations looks like the p-adic logarithm between distinct ring strs. "•";

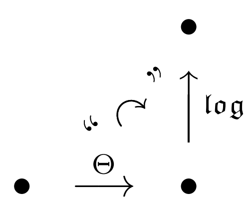
· the *descent operations* revolve around the establishment of certain *coricity/symmetry* properties.



$$(0,0) \overset{\text{(Stp1)}}{\rightsquigarrow} (0,\circ) \overset{\text{(Stp2)}}{\rightsquigarrow} (0,\circ)^{\vdash} \overset{\text{(Stp3)}}{\rightsquigarrow} (0,0)^{\vdash} \overset{\text{(Stp3)}}{\Longleftrightarrow} (1,0)^{\vdash}$$

— which involves a gradual introduction via "*descent*" operations of "*fuzzifications*", corresponding to *indeterminacies* [cf. the discussion of [EssLgc], §3.10].

· At a more technical level, the *multirad. rep. for the Θ-pilot* is obtained by constructing *invariants* with respects to the *$\mathfrak{log}$-link*, which has the effect of *juggling addition and multiplication* — i.e., juggling the *dilated* and *non-dilated* portions of the *ring strs.* — and, as a result, effects a sort of "*miraculous rotation*" (the discussion of [EssLgc], §3.11)



of the

· "*mysterious log-volume-dilating Θ-link gap*" (between the domain/codomain of the Θ-link) onto the

· "*harmless log-volume-preserving $\mathfrak{log}$-link gap*" (between the domain/codomain of the $\mathfrak{log}$-link) !

§3. <u>**Combinatorial function theory for idempotents**</u> (cf. [ArGT], §1)

- The following elementary result in *<u>combinatorial function theory</u>* concerning *<u>idempotents</u>* is the *<u>key technical lemma</u>* that underlies *<u>combinatorial algebraization theory (CAT)</u>*:

Let

- K be a perfect field;
- X an affine hyperbolic curve over K;
- $X^{\mathrm{act}} \subseteq X(\overline{K})$ an infinite subset;
- $Y \to X$ a finite étale Galois covering of hyperbolic curves over K, whose Galois group we denote by Q.

Write

$$R_X \subseteq R_Y \subseteq \mathrm{Fn}(Y^{\mathrm{act}}, \overline{K})$$

for the sub-K-algebras of the $\overline{K}$-algebra $\mathrm{Fn}(Y^{\mathrm{act}}, \overline{K})$ of (arbitrary set-theoretic) $\overline{K}$-valued functions on $Y^{\mathrm{act}} \stackrel{\mathrm{def}}{=} X^{\mathrm{act}} \times_{X(\overline{K})} Y(\overline{K})$ determined, via evaluation of functions at points $\in Y^{\mathrm{act}}$, by X and Y, respectively. Let A be a finite set and, for each $\alpha \in A$,

$$R_X \subseteq R_\alpha \subseteq \mathrm{Fn}(Y^{\mathrm{act}}, \overline{K})$$

a sub-K-algebra of $\mathrm{Fn}(Y^{\mathrm{act}}, \overline{K})$ that contains R_X, is isomorphic to R_Y as an R_X-algebra, and is stabilized by the natural action of Q on $\mathrm{Fn}(Y^{\mathrm{act}}, \overline{K})$ (induced by the natural action of Q on Y^{act}) in such a way that this natural action induces an isomorphism $Q \stackrel{\sim}{\to} \mathrm{Gal}(R_\alpha/R_X)$. Write

$$R_A \subseteq \mathrm{Fn}(Y^{\mathrm{act}}, \overline{K})$$

for the sub-R_X-algebra of $\mathrm{Fn}(Y^{\mathrm{act}}, \overline{K})$ generated by the sub-R_X-algebras R_α, as α ranges over the elements of A. Thus, one verifies immediately that, after possibly removing finitely many elements of X^{act} from X^{act}, we may assume without loss of generality that R_A is an R_X-*flat* quotient of the tensor product

$$R_\otimes \stackrel{\mathrm{def}}{=} \otimes_{\alpha \in A} R_\alpha$$

of R_X-algebras, hence is *finite étale* over R_X. In particular, $\mathrm{Spec}(R_A)$ has only finitely many connected components. Write

$$I$$

for the finite set of connected components of $\mathrm{Spec}(R_A)$;

$$E \stackrel{\mathrm{def}}{=} \{\epsilon_{i_0}\}_{i_0 \in I}$$

for the set of idempotents of R_A corresponding to the elements of I (i.e., idempotents whose support consists of precisely one connected component of $\mathrm{Spec}(R_A)$). Thus, Q acts naturally on R_A, $R_\otimes$, and E, and one verifies immediately that one may think of E as a set of mutually orthogonal idempotents of R_A of *maximal cardinality*. Then there exists an *infinite subset* $X^{\mathrm{act},\dagger} \subseteq X^{\mathrm{act}}$ such that, if we write

$$R_X^\dagger,\ R_Y^\dagger,\ R_A^\dagger,\ E^{\dagger,0}\ \subseteq\ \mathrm{Fn}(Y^{\mathrm{act},\dagger}, \overline{K})$$

for the subsets of the set $\mathrm{Fn}(Y^{\mathrm{act},\dagger}, \overline{K})$ of (arbitrary set-theoretic) $\overline{K}$-valued functions on $Y^{\mathrm{act},\dagger} \stackrel{\mathrm{def}}{=} X^{\mathrm{act},\dagger} \times_{X(\overline{K})} Y(\overline{K})$ determined, via evaluation of functions at points $\in Y^{\mathrm{act},\dagger}$, by R_X, R_Y, R_A and E, respectively, then the subset $E^\dagger \subseteq E^{\dagger,0}$ of *nonzero* elements of $E^{\dagger,0}$ satisfies the following properties:

(a) The set $E^\dagger$ is a set of mutually orthogonal idempotents of $R_A^\dagger$ of *maximal cardinality*, i.e., it is in *natural bijective correspondence* with the set of connected components of $\mathrm{Spec}(R_A^\dagger)$.

(b) The natural action of Q on $E^\dagger$ is *transitive*. In particular, the cardinality of $E^\dagger$ divides the order of Q.

(c) Let $H \subseteq Q$ be the stabilizer of an element of $E^\dagger$. (Thus, the Q-conjugacy class of H is independent of the choice of an element of $E^\dagger$.) Then for any $x \in X^{\mathrm{act},\dagger}$, the intersections with the Q-torsor

$$Y^{\mathrm{act},\dagger}|_x \stackrel{\mathrm{def}}{=} Y^{\mathrm{act},\dagger} \times_{X^{\mathrm{act},\dagger}} \{x\} \subseteq Y^{\mathrm{act},\dagger}$$

of the supports of the elements of $E^\dagger$ may be described, for a suitable choice of $y \in Y^{\mathrm{act},\dagger}|_x$, as the subsets $q \cdot H \cdot y \subseteq Y^{\mathrm{act},\dagger}|_x$, as q ranges over the elements of Q.

Proof (sketch) of (b):
Write

$$E_Q$$

for the set of Q-orbits of E. Thus, since E consists of mutually orthogonal idempotents, each element of E_Q may be thought of as a Q-invariant idempotent of R_A (i.e., by adding up the elements in the corresponding Q-orbit of E). Moreover, it follows immediately that the idempotents associated to elements of E_Q are also mutually orthogonal. Next, since the idempotents of E_Q are Q-invariant, it follows from the structure of $\mathrm{Fn}(Y^{\mathrm{act}}, \overline{K})$ that they may be thought of as idempotents of $\mathrm{Fn}(X^{\mathrm{act}}, \overline{K})$. Since E_Q is a finite set, by considering the supports of these idempotents of $\mathrm{Fn}(X^{\mathrm{act}}, \overline{K})$, we thus obtain a *finite partition* of X^{act}. Thus, since any finite partition of an infinite set as a finite disjoint union of subsets contains at least one infinite subset, we conclude that there exists a subset

$$X^{\mathrm{act},\dagger} \subseteq X^{\mathrm{act}}$$

of *infinite cardinality* such that $X^{\mathrm{act},\dagger}$ is the support in X^{act} of some element $\epsilon_Q \in E_Q$. In particular, it follows from the definition of $X^{\mathrm{act},\dagger}$ that an element $\epsilon \in E$ belongs to the Q-orbit ϵ_Q if and only if ϵ restricts to an element $\in E^\dagger \subseteq E^{\dagger,0}$. This equivalence immediately implies property (b).

§4. GT via combinatorial algebraization theory (CAT)
(cf. [CbGT]; [CbGal]; [ArGT]; [ArMCG])

- *Combinatorial algebraization theory (CAT)* may be understood as "*a Teichmüller theory*" analogous to
  - *$\mathbb{C}$Tch* (i.e., complex Teichmüller theory — cf. §1) and
  - *IUT* (cf. §2)

  for studying the conjugates of the image of

$$G_{\mathbb{Q}}, \qquad \Pi_{\mathcal{M}_{g,r/\mathbb{Q}}}$$

  inside

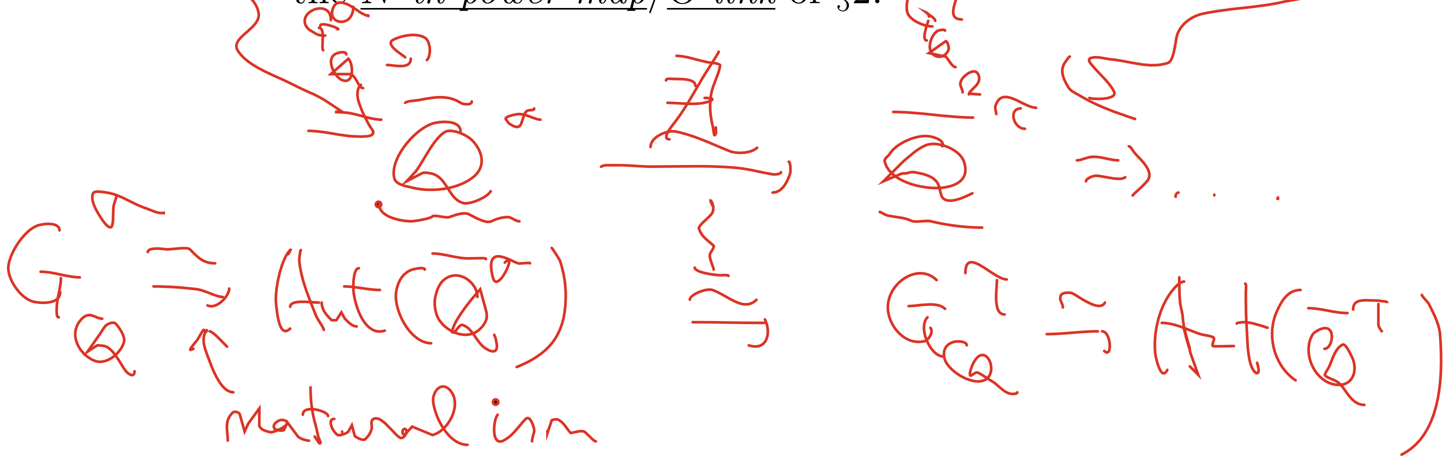
$$\mathrm{GT}\ (\subseteq \mathrm{Out}(\Pi_X)), \qquad \mathrm{Out}(\Pi_X)$$

  — where X denotes a hyperbolic curve over $\overline{\mathbb{Q}}$ of type $(0,3)$ or arbitrary (g,r) (such that $2g-2+r>0$).

- The starting point of CAT involves considering *distinct conjugates* of $G_{\mathbb{Q}}$ (or $\mathrm{Im}(\Pi_{\mathcal{M}_{g,r/\mathbb{Q}}})$) inside GT ($\ni \sigma,\tau$) (or $\mathrm{Out}(\Pi_X)$ ($\ni \sigma,\tau$))

$$\begin{array}{ccccc} G_{\mathbb{Q}}^{\sigma} & \subseteq & \mathrm{GT} & \supseteq & G_{\mathbb{Q}}^{\tau} \\ \mathrm{Im}(\Pi_{\mathcal{M}_{g,r/\mathbb{Q}}})^{\sigma} & \subseteq & \mathrm{Out}(\Pi_X) & \supseteq & \mathrm{Im}(\Pi_{\mathcal{M}_{g,r/\mathbb{Q}}})^{\tau} \end{array}$$

  ... that is to say, *distinct ring theories* that are related by a *mysterious non-ring-theoretic* — i.e., *purely combinatorial/ group-theoretic* — *link*, where the *non-algebraicity* of $\sigma\cdot\tau^{-1}$ plays the role of the *complex dilations* of §1, or, alternatively, the *N-th power map*/*Θ-link* of §2.



· In the case of CAT, the *analysis of idempotents* of §3 — i.e., roughly speaking,

$$K_X^{\sigma} \quad \subseteq \quad \prod_{\mathfrak{p}} \mathrm{Fn}(\{\text{pts. of } X \text{ over } \mathfrak{p}\}, \text{res. field of } \mathfrak{p}) \quad \supseteq \quad K_X^{\tau}$$

— where K_X denotes the function field of X, and $\mathfrak{p}$ runs over a Zariski dense set of finite primes of $\overline{\mathbb{Q}}$ or $\overline{\mathbb{Q}}$-points of $\mathcal{M}_{g,r/\mathbb{Q}}$ that arise from *Hurwitz schemes* — yields a "*miraculous rotation*" from the

- "*mysterious gap*" between the *distinct ring theories* associated to σ, τ-conjugates onto the
- "*harmless gap between Q-conjugates*" — where Q is a finite, center-free characteristic quotient of Π_X, and we note that such quotients are *cofinal* ! — provided by the *transitivity* property (b) of the Lemma of §3

... cf. the ring-th'ic interpretation of *Teichmüller maps* in §1; the *multiradial representation/"miraculous rotation"*

$$\Theta\text{-link} \text{ " } \curvearrowright \text{ " } \mathfrak{log}\text{-link}$$

of §2.

... Also, we note that just as in the case with the $\mathfrak{log}$-*link* of IUT — where we recall that forming *invariants* with respect to the $\mathfrak{log}$-link constitutes the *key step* in the construction of the *multirad. representation* of IUT — the "*miraculous rotation*" of CAT involves a rotation

$$\textit{addition} \text{ "}\boxplus\text{"} \qquad \text{" } \curvearrowright \text{ "} \qquad \textit{multiplication} \text{ "}\boxtimes\text{"},$$

i.e., at the level of ring structures between

- "$\oplus$" (cf. *distinct connected components* corresponding to *non-algebraic* $\sigma \cdot \tau^{-1}$!) and
- "$\otimes$" (cf. the construction of §3, which is closely related to the classical notion of the *Weil restriction*).

· One *important technical tool* in CAT is

 · *Chebotarev-type results ("Chb")* over *number fields*, as well as over *function fields over finite fields* (in the case of GT), and the
 · *Hilbertian property ("Hlb")* applied to *rational varieties* over *number fields* (in the case of $\mathrm{Out}(\Pi_X)$)

— which are used to compute the *multiplicative "⊗"* gp. actions (i.e., group actions on tensor products as in §3) since such group actions can only be related directly to the natural *additive "⊕" scheme-theoretic* group actions that arise naturally from Galois groups (or étale fundamental groups) in scheme theory at the *decomposition subgroups* associated to the points "$\mathfrak{p}$" appearing in the above product

$$\text{“}\prod_{\mathfrak{p}}\text{”}$$

(i.e., finite primes of $\overline{\mathbb{Q}}$ or $\overline{\mathbb{Q}}$-points of $\mathcal{M}_{g,r/\mathbb{Q}}$ arising from certain *Hurwitz schemes*).

· The *combinatorial anabelian geometry* developed in [CbGT], [CbGal] (especially, natural isomorphisms

$$\mathrm{Out}^{\mathrm{F}}(\Pi_{X_{n+1}}) \stackrel{\sim}{\to} \mathrm{Out}^{\mathrm{F}}(\Pi_{X_n})$$

of fiber subgroup-preserving outer autom. gps. of *configuration space groups* associated to X for sufficiently large n) — where we note that the passage $X_n \rightsquigarrow X_{n+1}$ corresponds, at "*toral*" (i.e., "*multiplicative*"!) nodes, to a passage to *tripods* (i.e., which involve "*additive*" symmetries $t \mapsto 1-t$), hence may be thought of as a sort of *combinatorial* analogue of the *p-adic logarithm*! (cf. the vertical columns of $\mathfrak{log}$-links in IUT!) — constitutes another *crucial technical tool* in CAT
. . . cf. the fundamental role played throughout IUT by
 $\mathfrak{log}$-links, absolute p-adic anabelian geometry!

· The technique of *cyclotomic combinatorial Belyi cuspidalizations* (cf. [CbGal])

$$\begin{array}{ccc} U & \hookrightarrow & X \\ \big\downarrow {\scriptstyle (-)^N} & & \\ X & & \end{array}$$

also plays an *important role* in CAT
. . . cf. the fundamental role played throughout IUT by
 the *étale theta fn./elliptic cuspidalizations*!

· The theory discussed above may be summarized as follows:

$\mathbb{C}$Tch	IUT	CAT for GT, Out(Π_X)
distinct holomorphic structures $\mathcal{O}^{\mathfrak{H}_i}$, for $i = 1, 2$, on **same** underlying top. surface	**distinct ring/arith. hol. structures** on opposite sides of the **Θ-link**	**distinct ring structures** corresponding to distinct **conjugates** of $G_{\mathbb{Q}}$ or $\Pi_{\mathcal{M}_{g,r/\mathbb{Q}}}$ inside GT or Out(Π_X)
embedding of $\mathcal{O}^{\mathfrak{H}_i}$'s into **common container/ domain** $\mathcal{O}^{\mathfrak{R}}$ via **Teichmüller maps**	**multiradial rep.**, up to *mild indets.*, yields **common container** for distinct ring/arith. hol. strs. via **miraculous rotation** Θ-link " $\curvearrowright$ " $\mathfrak{log}$-link involving $\mathfrak{log}$-*invars.* via a rotation $\boxtimes$ " $\curvearrowright$ " $\boxplus$, **absolute *p*-adic anab. geo.** (cf., especially, (Ind3)), **étale theta fn./ elliptic cuspidalizations**	**embedding** of distinct ring strs. into $\prod_{\mathfrak{p}}$ Fn(−, −) via analysis of idems./Chb/Hlb yields a **miraculous rotation** GT, Out(Π_X)-cnjs. " $\curvearrowright$ " Q-cnjs. via a rotation $\otimes$ " $\curvearrowright$ " $\oplus$, **combinatorial anab. geo.**, **cyclotomic arithmetic Belyi cuspidalizations**

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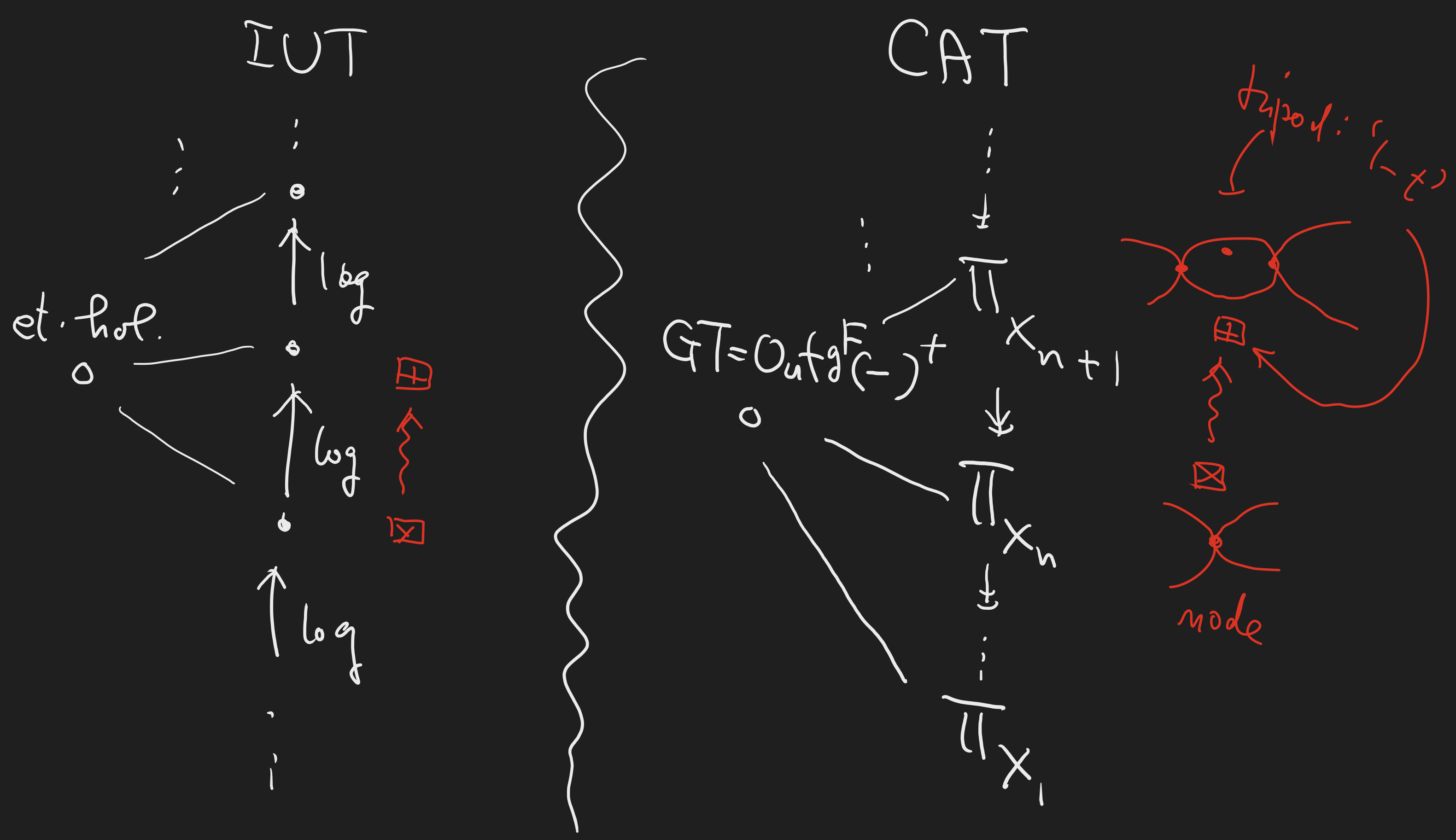

Ind3 hull <- - - -> $GT \subseteq Out(\Pi_X)$

(⊢)

ell. cusp. <- - - - -> cycl.cb. Belyi. cusp.